

On Certain Results Involving Bilateral Basic Hypergeometric Functions and Ratio of Infinite Products

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Abstract:- Some bilateral basic hypergeometric functions can be expressed in the form of ratio of infinite products.

In this paper, an attempt has been made to establish the results involving two bilateral basic hypergeometric functions and ratio of infinite products. We have also established the results involving three bilateral basic hypergeometric functions and ratio of infinite products by using suitable identities and changing the parameters in well known results involving bilateral basic hypergeometric functions.

Key words:- bilateral basic hypergeometric functions , ratio of infinite products.

1. Introduction

By using identities, Bindu Prakash Mishra [1] established transformations involving bilateral basic Hypergeometric Functions.

$$\begin{aligned}
 (1.1) \quad & (a - \alpha q/a) \frac{[\alpha q/ab, -\alpha q/b, -q/b; q]_{\infty}}{[-\alpha q^2/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\
 & = a(1 - \alpha q/ab) \frac{[\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q/a, q/a; q]_{\infty}} \\
 & \quad - \frac{\alpha q(1-a)[-\alpha q^2, \alpha/ab, -1/b; q]_{\infty}}{a[-\alpha q/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} aq, -bq; q; \alpha/ab \\ \alpha q^2/a, \alpha q^2/b \end{matrix} \right]
 \end{aligned}$$

$$\begin{aligned}
 (1.2) \quad & (a - \alpha q/a) \frac{[-\alpha q/b, \alpha q/ab, -q/b; q]_{\infty}}{[-\alpha q/ab, -\alpha q^2/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\
 & = \frac{a[\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, -\alpha q/ab, q/a; q]_{\infty}} \\
 & \quad - \frac{\alpha q(1-1/a)[-\alpha q^2/b, -\alpha q/a, -q/b, -1/b; q]_{\infty}}{[\alpha q/b, 1/a, \alpha q^2/a; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a, -bq; q; -\alpha q/ab \\ -\alpha q^2/b, -\alpha q/a \end{matrix} \right]
 \end{aligned}$$

$$(1.3) \quad (a - \alpha q/a) [-\alpha q/a, -\alpha q/b, -q^2/a, -q/b; q]_{\infty} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^2/ab \\ -\alpha q/a, -\alpha q/b \end{matrix} \right]$$

$$= \frac{a[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[-\alpha q/ab; q]_{\infty}} + \alpha q[-\alpha q^2/b, -\alpha q/a, -q/a, -1/b; q]_{\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -bq; q; -\alpha q/ab \\ -\alpha q^2/b, -\alpha q/a \end{matrix} \right] \text{ and so on.}$$

Here an attempt has been made to establish the results involving two bilateral basic Hypergeometric Functions and ratio of infinite products. We have also established the results involving three bilateral basic Hypergeometric Functions and ratio of infinite products by using suitable identities and changing the parameters in well known results involving bilateral basic Hypergeometric Functions.

2. Notation

Consider $|q| < 1$, where q is non-zero complex number, this condition ensures all the infinite products that we use will converge. We will use the notation,

$$(2.1) \quad [a; q]_n = \begin{cases} (1 - a)(1 - aq) \dots \dots (1 - aq^{n-1}); & n > 0 \\ 1 & n = 0, \end{cases}$$

$$(2.2) \quad [a; q]_{\infty} = \prod_{r=0}^{\infty} (1 - aq^r),$$

$$(2.3) \quad [a; q]_{-n} = \frac{(-)^n q^{n(n+1)/2}}{\alpha^n [a/q; q]_n},$$

$$(2.4) \quad [z_1, z_2, \dots \dots \dots z_n; q]_{\infty} = [z_1; q]_{\infty} [z_2; q]_{\infty} \dots \dots \dots [z_n; q]_{\infty},$$

$$(2.5) \quad [z, zq; q^2]_{\infty} = [z; q]_{\infty}$$

Following the above notation, we define

$$(2.6) \quad {}_r\Phi_s \left[\begin{matrix} a_1, a_2, a_3, \dots, a_r; q; z \\ b_1, b_2, \dots, b_s \end{matrix} \right] = \sum_{n=0}^{\infty} \frac{[a_1, a_2, a_3, \dots, a_r; q]_n z^n}{[q, b_1, b_2, \dots, b_s]_n}$$

Max($|q|, |z| < 1$). where

$$[a_1, a_2 \dots \dots a_r; q]_n = [a_1; q]_n [a_2; q]_n \dots \dots [a_r; q]_n$$

Also, we define the Basic Bilateral Hypergeometric Function

$$(2.7) \quad {}_r\Psi_r \left[\begin{matrix} a_1, a_2, \dots, a_r; q; z \\ b_1, \dots, b_r \end{matrix} \right] = \sum_{n=-\infty}^{\infty} \frac{[a_1, a_2, \dots, a_r; q]_n z^n}{[b_1, b_2, \dots, b_r]_n}$$

where for convergence $|b_1, b_2, \dots \dots b_r/a_1, a_2 \dots \dots a_r| < z < 1$.

We shall make use of the following known results

$$(2.8) \quad {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] = \frac{[az, d/a, c/b, dq/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, c \end{matrix} \right]$$

[Gasper-Rahman 1; (5.20 (i) p. 137]

$$(2.9) \quad {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] = \frac{[az, bz, cq/abz, dq/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c, abz/d; q; cd/abz \\ az, bz \end{matrix} \right]$$

[Gasper-Rahman 1; (5.20 (ii) p. 137]

$$(2.10) \quad {}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q/a, \alpha q/b, q/a, q/b, -\alpha q/ab; q]_\infty}$$

[Gasper-Rahman 1; App.II (II.30 p. 239]

3. Main Result

$$(3.1) \quad \frac{\alpha(1-a)}{aq(1-aq)} \frac{[\alpha/a^2, \alpha/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha/a, 1/a; q]_\infty} = \frac{[-\alpha/bq, \alpha/ab, -q^2/b; q]_\infty}{[-\alpha/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -\alpha/q^2; q; \alpha/ab \\ -\alpha/bq, \alpha/aq \end{matrix} \right]$$

$$- \frac{\alpha}{bq} \frac{[-\alpha/b, \alpha/ab, -q/b; q]_\infty}{[-\alpha/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha/ab \\ -\alpha/b, \alpha/aq \end{matrix} \right]$$

$$(3.2) \quad \frac{\alpha}{q(\frac{\alpha}{aq}-1)} \frac{[\alpha/abq; q]_\infty [\alpha/a^2, \alpha/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha/a, q/a, -\alpha/abq; q]_\infty}$$

$$= \frac{[-\alpha/bq, \alpha/ab, \alpha/abq, -q^2/b; q]_\infty}{[-\alpha/abq, -\alpha/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha/ab \\ -\alpha/bq, \alpha/aq \end{matrix} \right]$$

$$- \frac{\alpha}{bq} \frac{[-\alpha/b, -\alpha/a, -1/a, -q/b; q]_\infty}{[q/a, \alpha/aq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -aq, -b; q; -\alpha/abq \\ -\alpha/b, -\alpha/a \end{matrix} \right]$$

$$(3.3) \quad \frac{\alpha}{q(\alpha/aq-1)} \frac{[\alpha/abq; q]_\infty [\alpha/a^2, \alpha/b^2, q^2, -\alpha/q, q/\alpha; q^2]_\infty}{[\alpha/a, -\alpha/abq; q]_\infty}$$

$$= \frac{[-\alpha/bq, -\alpha/aq, -q/a, -q^2/b; q]_\infty}{[\alpha/aq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b/q; q; -\alpha/ab \\ -\alpha/bq, -\alpha/aq \end{matrix} \right]$$

$$- \frac{\alpha}{bq} \frac{[-\alpha/b, -\alpha/a, -1/a, -q/b; q]_\infty}{[\alpha/aq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -aq, -b/q; q; -\alpha/abq \\ -\alpha/b, -\alpha/a \end{matrix} \right]$$

$$(3.4) \quad \frac{\alpha(1-a)}{aq(1-aq)} \frac{[\alpha/abq; q]_\infty [\alpha/a^2, \alpha/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha/a, 1/a, -\alpha/abq; q]_\infty}$$

$$= \frac{[-\alpha/bq, -\alpha/aq, -q/a, -q^2/b; q]_\infty}{[q/a, \alpha/aq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b/q; q; -\alpha/ab \\ -\alpha/bq, -\alpha/aq \end{matrix} \right]$$

$$(3.5) \quad \frac{\alpha \frac{[-\alpha/b, \alpha/ab, \alpha/abq, -q/b; q]_{\infty}}{bq} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha/ab \\ -\alpha/b, \alpha/aq \end{matrix} \right]}{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}} = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_{\infty}}{[-\alpha q/ab, -\alpha q^3/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2 ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right]$$

$$(3.6) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[-\alpha q/ab; q]_{\infty}} = \frac{-\frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_{\infty}}{[-\alpha q^2/ab, -\alpha q^2/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right]}{+\frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_{\infty}}{[-\alpha q^3/ab, -\alpha q/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]}$$

$$(3.6) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[-\alpha q/ab; q]_{\infty}} = \frac{[-\alpha q/a, -\alpha q/b, -q^2/a, -q^2/b; q]_{\infty}}{[-\alpha q/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right]$$

$$(3.7) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[-\alpha q/ab; q]_{\infty}} = \frac{-\frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_{\infty}}{[-\alpha q^2/b, -\alpha q^2/a]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right]}{+\frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_{\infty}}{[-\alpha q^3/b, -\alpha q^3/a]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]}$$

$$(3.7) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_{\infty}} = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_{\infty}}{[-\alpha q/ab, -\alpha q^3/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right]$$

$$(3.7) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_{\infty}} = \frac{-\frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_{\infty}}{[-\alpha q^2/ab, -\alpha q^2/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right]}{+\frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_{\infty}}{[q/a, \alpha q^2/a; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]}$$

$$(3.8) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_{\infty}} = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_{\infty}}{[q/a, \alpha q^2/a; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right]$$

$$(3.8) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_{\infty}} = \frac{-\frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_{\infty}}{[-\alpha q^2/ab, -\alpha q^2/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right]}{+\frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_{\infty}}{[-\alpha q^3/ab, -\alpha q/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]}$$

$$(3.9) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_{\infty}} = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_{\infty}}{[-\alpha q/ab, -\alpha q^3/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right]$$

$$(3.9) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_{\infty}} = \frac{-\frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_{\infty}}{[q/a, \alpha q^2/a; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right]}{+\frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_{\infty}}{[-\alpha q^3/ab, -\alpha q/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -bq^2; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]}$$

$$(3.10) \quad \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_{\infty}} = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_{\infty}}{[q/a, \alpha q^2/a; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right]$$

$$\begin{aligned}
 & - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_\infty}{[-\alpha q^2/ab, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right] \\
 & + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -aq, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right] \\
 (3.11) \quad & \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q]_\infty}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_\infty} = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right] \\
 & - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right] \\
 & + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_\infty}{[-\alpha q^3/ab, -\alpha q/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]
 \end{aligned}$$

$$\begin{aligned}
 (3.12) \quad & \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q^2/a, \alpha q^2/b, q/a, -\alpha q/ab; q]_\infty} = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_\infty}{[-\alpha q/ab, -\alpha q^3/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\
 & - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_\infty}{[q/a, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right] \\
 & + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -aq, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]
 \end{aligned}$$

4.Proof

To establish main results (3.1) to (3.4), consider a suitable identity

$$(1-a)(1-b) = \left(\frac{1-a-b+ab}{1-dq^n} \right) - \left(\frac{1-a-b+ab}{(1-dq^n)} \right) dq^n$$

$$(4.1) \quad \frac{z(1-a)(1-b)}{(1-c)} = \frac{z(1-a)(1-b)}{(1-c)(1-dq^n)} - \frac{z(1-a)(1-b)dq^n}{(1-c)(1-dq^n)}$$

Multiplying by $\frac{(aq)_n(bq)_n}{(cq)_n(d)_n} z^n$ both side of (4.1) and summing over from $-\infty$ to ∞ , we get

$$(4.2) \quad \frac{z(1-a)(1-b)}{(1-c)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; z \\ cq, d, \end{matrix} \right] = {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d, \end{matrix} \right] - \frac{d}{q} {}_2\Psi_2 \left[\begin{matrix} a, b; q; zq \\ c, d, \end{matrix} \right]$$

Again from (2.10)

Replacing a by aq , b by bq in (2.10), we get

$$(4.3) \quad {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; -\alpha/abq \\ \alpha/a, \alpha/b; \end{matrix} \right] = \frac{[\alpha/abq; q]_\infty [\alpha/a^2, \alpha/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha/a, \alpha/b, 1/a, 1/b, -\alpha/abq; q]_\infty}$$

Now, by transforming each ${}_2\Psi_2$ on right side of (4.2) with the help of (2.8), we get

$$(4.4) \quad \frac{z(1-a)(1-b)}{(1-c)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; z \\ cq, d, \end{matrix} \right] = \frac{[az, d/a, c/b, dq/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, c; \end{matrix} \right]$$

$$-\frac{d}{q} \frac{[azq, d/a, c/b, d/abz; q]_{\infty}}{[zq, d, q/b, cd/abzq; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, abzq/d; q; d/a \\ azq, c, \end{matrix} \right]$$

Further replacing z by $-\alpha/abq$, c by α/aq , and d by α/b in (4.4)

$$\frac{(-\alpha/abq)(1-a)(1-b)}{(1-\alpha/aq)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; -\alpha/abq \\ \alpha/a, \alpha/b; \end{matrix} \right] = \frac{[-\alpha/bq, \alpha/ab, \alpha/abq, -q^2/b; q]_{\infty}}{[-\alpha/abq, \alpha/b, q/b, -\alpha/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -\alpha/q^2; q; \alpha/ab \\ -\alpha/bq, \alpha/aq; \end{matrix} \right]$$

$$-\frac{\alpha}{bq} \frac{[-\alpha/b, \alpha/ab, \alpha/abq, -q/b; q]_{\infty}}{[-\alpha/ab, \alpha/b, q/b, -\alpha/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha/ab \\ -\alpha/b, \alpha/aq \end{matrix} \right]$$

$$(4.5) \frac{\alpha(1-a)(1-b)}{abq(\alpha/aq-1)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; -\alpha/abq \\ \alpha/a, \alpha/b \end{matrix} \right] = \frac{[-\alpha/bq, \alpha/ab, \alpha/abq, -q^2/b; q]_{\infty}}{[-\alpha/abq, \alpha/b, q/b, -\alpha/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -\alpha/q^2; q; \alpha/ab \\ -\alpha/bq, \alpha/aq \end{matrix} \right]$$

$$-\frac{\alpha}{bq} \frac{[-\alpha/b, \alpha/ab, \alpha/abq, -q/b; q]_{\infty}}{[-\alpha/ab, \alpha/b, q/b, -\alpha/abq; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha/ab \\ -\alpha/b, \alpha/aq \end{matrix} \right]$$

By using (4.3) in L.H.S. of (4.5), we get

$$(4.6) \frac{\alpha(1-a)}{aq(1-\alpha/aq)} \frac{[\alpha/a^2, \alpha/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha/a, 1/a; q]_{\infty}} = \frac{[-\alpha/bq, \alpha/ab, -q^2/b; q]_{\infty}}{[-\alpha/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -\alpha/q^2; q; \alpha/ab \\ -\alpha/bq, \alpha/aq \end{matrix} \right]$$

$$-\frac{\alpha}{bq} \frac{[-\alpha/b, \alpha/ab, -q/b; q]_{\infty}}{[-\alpha/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha/ab \\ -\alpha/b, \alpha/aq \end{matrix} \right]$$

This is our main result (3.1)

Now, transforming $Ist {}_2\Psi_2$ with the help of (2.8) and $II^{nd} {}_2\Psi_2$ with the help of (2.9) in R.H.S. of (4.2)

$$(4.7) \frac{z(1-a)(1-b)}{(1-c)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; z \\ cq, d \end{matrix} \right] = \frac{[az, d/a, c/b, dq/abz; q]_{\infty}}{[z, d, q/b, cd/abz; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; qd/a \\ az, c \end{matrix} \right]$$

$$-\frac{d}{q} \frac{[azq, bzq, c/abz, d/abz; q]_{\infty}}{[q/a, q/b, c, d, q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} abzq/c, abzq/d; q; cd/abzq \\ azq, bzq \end{matrix} \right]$$

Taking $z = -\alpha/abq$, $c = \alpha/aq$, & $d = \alpha/b$ in (4.7)

$$(4.8) \frac{\frac{-\alpha}{abq}(1-a)(1-b)}{(1-\alpha/aq)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; -\alpha/abq \\ \alpha/a, \alpha/b \end{matrix} \right] = \frac{[-\alpha/bq, \alpha/ab, \alpha/abq, -q^2/b; q]_{\infty}}{[-\alpha/abq, \alpha/b, q/b, -\alpha/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha/ab \\ -\alpha/bq, \alpha/aq \end{matrix} \right]$$

$$-\frac{\alpha}{bq} \frac{[-\alpha/b, -\alpha/a, -1/a, -q/b; q]_{\infty}}{[q/a, q/b, \alpha/aq, \alpha/b; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -aq, -b; q; -\alpha/abq \\ -\alpha/b, -\alpha/a \end{matrix} \right]$$

By using (4.3) in LHS of (4.8), we get

$$(4.9) \frac{\alpha}{q(\frac{\alpha}{aq}-1)} \frac{[\alpha/abq; q]_{\infty} [\alpha/a^2, \alpha/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha/a, q/a - \alpha/abq; q]_{\infty}}$$

$$= \frac{[-\alpha/bq, \alpha/ab, \alpha/abq, -q^2/b; q]_{\infty}}{[-\alpha/abq, -\alpha/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha/ab \\ -\alpha/bq, \alpha/aq \end{matrix} \right]$$

$$-\frac{\alpha}{bq} \frac{[-\alpha/b, -\alpha/a, -1/a, -q/b; q]_{\infty}}{[q/a, \alpha/aq; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -aq, -b; q; -\alpha/abq \\ -\alpha/b, -\alpha/a \end{matrix} \right]$$

This is our main result (3.2)

Now transforming each ${}_2\Psi_2$ on right side of (4.2) with the help of (2.9)

$$(4.10) \frac{z(1-a)(1-b)}{(1-c)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; z \\ cq, d; \end{matrix} \right] = \frac{[az, bz, cq/abz, dq/abz; q]_\infty} {[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c; abz/d; q; cd/abz \\ az, bz \end{matrix} \right] \\ - \frac{d [azq, bzq, c/abz, d/abz; q]_\infty}{q [q/a, q/b, c, d, q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abzq/c, abzq/d; q; cd/abzq \\ azq, bzq \end{matrix} \right]$$

Taking $z = -\alpha/abq, c = \alpha/aq, & d = \alpha/b$ in (4.10) on both side

$$(4.11) \frac{\frac{-\alpha}{abq}(1-a)(1-b)}{(1-\alpha/aq)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; -\alpha/abq \\ \alpha/a, \alpha/b \end{matrix} \right] \\ = \frac{[-\alpha/bq, -\alpha/aq, -q/a, -q^2/b; q]_\infty}{[q/a, q/b, \alpha/aq, \alpha/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b/q; q; -\alpha/ab \\ -\alpha/bq, -\alpha/aq \end{matrix} \right] \\ - \frac{\alpha [-\alpha/b, -\alpha/a, -1/a, -q/b; q]_\infty}{bq [q/a, q/b, \alpha/aq, \alpha/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -aq, -b; q; -\alpha/abq \\ -\alpha/b, -\alpha/a \end{matrix} \right]$$

By using (4.3) in L.H.S. of (4.11)

$$(4.12) \frac{\alpha}{q(\alpha/aq-1)} \frac{[\alpha/abq; q]_\infty [\alpha/a^2, \alpha/b^2, q^2, -\alpha/q, q/\alpha; q^2]_\infty}{[\alpha/a, -\alpha/abq; q]_\infty} \\ = \frac{[-\alpha/bq, -\alpha/aq, -q/a, -q^2/b; q]_\infty}{[\alpha/aq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b/q; q; -\alpha/ab \\ -\alpha/bq, -\alpha/aq \end{matrix} \right] \\ - \frac{\alpha [-\alpha/b, -\alpha/a, -1/a, -q/b; q]_\infty}{bq [\alpha/aq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -aq, -b/q; q; -\alpha/abq \\ -\alpha/b, -\alpha/a \end{matrix} \right]$$

This is our main result (3.3)

Now transforming the $\Gamma^{\text{st}} {}_2\Psi_2$ with the help of (2.9) and $\Pi^{\text{nd}} {}_2\Psi_2$ with the help of (2.8) in R.H.S. of (4.2), we get

$$(4.13) \frac{z(1-a)(1-b)}{(1-c)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; z \\ cq, d; \end{matrix} \right] = \frac{[az, bz, cq/abz, dq/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c; abz/d; q; cd/abz \\ az, bz \end{matrix} \right] \\ - \frac{d [azq, d/a, c/b, d/abz; q]_\infty}{q [zq, d, q/b; cd/abzq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abzq/d; q; d/a \\ azq, c \end{matrix} \right]$$

Taking $z = -\alpha/abq, c = \alpha/aq, & d = \alpha/b$ in (4.13)

$$(4.14) \frac{(-\alpha/abq)(1-a)(1-b)}{(1-\alpha/aq)} {}_2\Psi_2 \left[\begin{matrix} aq, bq; q; -\alpha/abq \\ \alpha/a, \alpha/b; \end{matrix} \right]$$

$$= \frac{[-\alpha/bq, -\alpha/aq, -q/a, -q^2/b; q]_\infty}{[q/a, q/b, \alpha/aq, \alpha/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b/q; q; -\alpha/ab \\ -\alpha/bq, -\alpha/aq \end{matrix} \right] \\ - \frac{\alpha}{bq} \frac{[-\alpha/b, \alpha/ab, \alpha/abq, -q/b; q]_\infty}{[-\alpha/ab, \alpha/b, q/b, -\alpha/abq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha/ab \\ -\alpha/b, \alpha/aq; \end{matrix} \right]$$

Now by using (4.3) in RHS of (4.14)

$$(4.15) \quad \frac{\alpha(1-a)}{aq(1-\alpha/aq)} \frac{[\alpha/abq; q]_\infty [\alpha/a^2, \alpha/b^2, q^2, \alpha q, q/a; q^2]_\infty}{[\alpha/a, 1/a, -\alpha/abq; q]_\infty} \\ = \frac{[-\alpha/bq, -\alpha/aq, -q/a, -q^2/b; q]_\infty}{[q/a, \alpha/aq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b/q; q; -\alpha/ab \\ -\alpha/bq, -\alpha/aq \end{matrix} \right] \\ - \frac{\alpha}{bq} \frac{[-\alpha/b, \alpha/ab, \alpha/abq, -q/b; q]_\infty}{[-\alpha/ab, -\alpha/abq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha/ab \\ -\alpha/b, \alpha/aq \end{matrix} \right]$$

This is our main result (3.4)

To establish main results (3.5) to (3.12), let us consider the following suitable identity.

$$(4.16) \quad (1 - cq^n)(1 - dq^n) = 1 - (c + d)q^n + cdq^{2n}$$

Multiplying by $\frac{(a)_{n+1}(b)_{n+1}}{(c)_n(d)_n} z^{n+1}$ on both sides of (4.6) we get

$$\frac{(a)_{n+1}(b)_{n+1}}{(c)_n(d)_n} z^{n+1} = \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(d)_{n+1}} z^{n+1} - \left(\frac{c+d}{q}\right) \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(d)_{n+1}} (zq)^{n+1} + \frac{cd}{q^2} \frac{(a)_{n+1}(b)_{n+1}}{(c)_{n+1}(d)_{n+1}} (zq^2)^{n+1}$$

By summing the above over m from $-\infty$ to ∞ , we have

$$\left(1 - \frac{c}{q}\right) \left(1 - \frac{d}{q}\right) \sum_{m=-\infty}^{\infty} \frac{(a)_m(b)_m}{(c/q)_m(d/q)_m} z^m = \sum_{m=-\infty}^{\infty} \frac{(a)_m(b)_m}{(c)_m(d)_m} z^m - \left(\frac{c+d}{q}\right) \sum_{m=-\infty}^{\infty} \frac{(a)_m(b)_m}{(c)_m(d)_m} (zq)^m \\ + \frac{cd}{q^2} \sum_{m=-\infty}^{\infty} \frac{(a)_m(b)_m(zq^2)^m}{(c)_m(d)_m}$$

$$(4.17) \quad (1-c/q)(1-d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] = {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c, d \end{matrix} \right] - \frac{c+d}{q} {}_2\Psi_2 \left[\begin{matrix} a, b; q; zq \\ c, d \end{matrix} \right] + \frac{cd}{q^2} {}_2\Psi_2 \left[\begin{matrix} a, b; q; zq^2 \\ c, d \end{matrix} \right]$$

Now by transforming each ${}_2\Psi_2$ on right side of (4.17) with the help of (2.8)

$$(4.18) \quad (1 - c/q)(1 - d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] = \frac{[az, d/a, c/b, dq/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, c \end{matrix} \right] \\ - \left(\frac{c+d}{q}\right) \frac{[azq, d/a, c/b, d/abz; q]_\infty}{[zq, d, q/b, cd/abzq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abzq/d; q; d/a \\ azq, c \end{matrix} \right]$$

$$+ \frac{cd}{q^2} \frac{[azq^2, d/a, c/b, d/abzq; q]_\infty}{[zq^2, d, q/b, cd/abzq^2; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abzq^2/d; q; d/a \\ azq^2, c \end{matrix} \right]$$

Taking $c = \alpha q^2/a, d = \alpha q^2/b$ and $z = -\alpha q/ab$ in (4.18), we get

$$(4.19) \quad (1 - \alpha q/a)(1 - \alpha q/b) {}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_\infty}{[-\alpha q/ab, \alpha q^2/b, q/b, -\alpha q^3/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_\infty}{[-\alpha q^2/ab, \alpha q^2/b, q/b, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right] \\ + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_\infty}{[-\alpha q^3/ab, \alpha q^2/b, q/b, -\alpha q/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]$$

By using (2.10) in L.H.S of (4.19) .

$$(4.20) \quad \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q^2/a, q/\alpha - \alpha q/ab; q]_\infty} = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_\infty}{[-\alpha q/ab, -\alpha q^3/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2 ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_\infty}{[-\alpha q^2/ab, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right] \\ + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_\infty}{[-\alpha q^3/ab, -\alpha q/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]$$

This is our main result (3.5)

Now by transforming each ${}_2\Psi_2$ on right side of (4.17) with the help of (2.9)

$$(4.21) \quad (1 - c/q)(1 - d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] = \frac{[az, bz, cq/abz, dq/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c, abz/d; q; cd/abz \\ az, bz \end{matrix} \right] \\ - \left(\frac{c+d}{q} \right) \frac{[azq, bzq, c/abz, d/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abzq/c, abzq/d; q; cd/abzq \\ azq, bzq \end{matrix} \right] \\ + \frac{cd}{q^2} \frac{[azq^2, bzq^2, c/abzq, d/abzq; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abzq^2/c, abzq^2/d; q; cd/abzq^2 \\ azq^2, bzq^2 \end{matrix} \right]$$

Taking $c = \alpha q^2/a, d = \alpha q^2/b$, and $z = -\alpha q/ab$ in (4.21), we get

$$(4.22) \quad (1 - \alpha q/a)(1 - \alpha q/b) {}_2\Psi_2 \left[\begin{matrix} a, b, q, -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[-\alpha q/a, -\alpha q/b, -q^2/a; -q^2/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right]$$

$$\begin{aligned}
 & - \frac{\alpha(a+b)q}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right] \\
 & + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]
 \end{aligned}$$

By using (2.10) in L.H.S. of (4.22)

$$\begin{aligned}
 (4.23) \quad \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[-\alpha q/ab; q]_\infty} &= [-\alpha q/a, -\alpha q/b, -q^2/a, -q^2/b; q]_\infty {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right] \\
 & - \frac{\alpha q(a+b)}{ab} [-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_\infty {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right] \\
 & + \frac{\alpha^2 q^2}{ab} [-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]
 \end{aligned}$$

This is our main result (3.6).

Now by transforming I^{st} & II^{nd} ${}_2\Psi_2$ with the help of (2.8) and III^{rd} ${}_2\Psi_2$ with the help of (2.9), in right side of (4.17), we get

$$\begin{aligned}
 (4.24) \quad (1 - c/q)(1 - d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] &= \frac{[az, d/a, c/b, dq/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, c \end{matrix} \right] \\
 & - \frac{c+d}{q} \frac{[azq, d/a, c/b, d/abz; q]_\infty}{[zq, d, q/b, cd/abzq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abzq/d; q; d/a \\ azq, c \end{matrix} \right] \\
 & + \frac{cd}{q^2} \frac{[azq^2, bzq^2, c/abzq, d/abzq; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abzq^2/c, abzq^2/d; q; cd/abzq^2 \\ azq^2, bzq^2 \end{matrix} \right]
 \end{aligned}$$

Taking $c = \alpha q^2/a, d = \alpha q^2/b$, and $z = -\alpha q/ab$ in (4.24)

$$\begin{aligned}
 (4.25) \quad (1 - \alpha q/a)(1 - \alpha q/b) {}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] &= \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_\infty}{[-\alpha q/ab, \alpha q^2/b, q/b, -\alpha q^3/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\
 & - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_\infty}{[-\alpha q^2/ab, \alpha q^2/b, q/b, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right] \\
 & + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]
 \end{aligned}$$

By using (2.10) in L.H.S. of (4.25)

$$\begin{aligned}
 (4.26) \quad \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_\infty} &= \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_\infty}{[-\alpha q/ab, -\alpha q^3/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\
 & - \frac{\alpha q(a+b)}{ab} {}_2\Psi_2 \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_\infty}{[-\alpha q^2/ab, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right]
 \end{aligned}$$

$$+ \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]$$

This is our main result (3.7)

Now by transforming 1^{st} ${}_2\Psi_2$ with the help of (2.9) and II^{nd} & III^{rd} ${}_2\Psi_2$ with the help of (2.8), in right side of (4.17), we get

$$(4.27) \quad (1 - c/q)(1 - d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] = \frac{[az, bz, cq/abz, dq/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c; abz/d; q; cd/abz \\ az, bz \end{matrix} \right] \\ - \frac{c+d}{q} \frac{[azq, d/a, c/b, d/abz; q]_\infty}{[zq, d, q/b, cd/abzq; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abzq/d; q; d/a \\ azq, c \end{matrix} \right] \\ + \frac{cd}{q^2} \frac{[azq^2, d/a, c/b, d/abzq; q]_\infty}{[zq^2, d, q/b, cd/abzq^2; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abzq^2/d; q; d/a \\ azq^2, c \end{matrix} \right]$$

Taking $c = \alpha q^2/a, d = \alpha q^2/b$, and $z = -\alpha q/ab$, in (4.27) we get

$$(4.28) \quad (1 - \alpha q/a)(1 - \alpha q/b) {}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_\infty}{[-\alpha q^2/ab, \alpha q^2/b, q/b, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right] \\ + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_\infty}{[-\alpha q^3/ab, \alpha q^2/b, q/b, -\alpha q/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]$$

By using (2.10) in L.H.S. of (4.28), we get

$$(4.29) \quad \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_\infty} = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_\infty}{[-\alpha q^2/ab, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right] \\ + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_\infty}{[-\alpha q^3/ab, -\alpha q/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]$$

This is our main result (3.8)

Now by transforming 1^{st} & III^{rd} ${}_2\Psi_2$ with the help of (2.8) and II^{nd} ${}_2\Psi_2$ with the help of

(2.9), in right side of (4.17), we get

$$(4.30) \quad (1 - c/q)(1 - d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] = \frac{[az, d/a, c/b, dq/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, c \end{matrix} \right]$$

$$-\frac{c+d}{q} \frac{[azq, bzq, c/abz, d/abz; q]_{\infty}}{[q/a, q/b, c, d; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} abzq/c, abzq/d; q; cd/abzq \\ azq, bzq \end{matrix} \right]$$

$$+\frac{cd}{q^2} \frac{[azq^2, d/a, c/b, d/abzq; q]_{\infty}}{[zq^2, d, q/b, cd/abzq^2; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, abzq^2/d; q; d/a \\ azq^2, c \end{matrix} \right]$$

Taking $c = \alpha q^2/a, d = \alpha q^2/b$, & $z = -\alpha q/ab$, in (4.30)

$$(4.31) (1 - \alpha q/a)(1 - \alpha q/b) {}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_{\infty}}{[-\alpha q/ab, \alpha q^2/b, q/b, -\alpha q^3/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right]$$

$$-\frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_{\infty}}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right]$$

$$+\frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_{\infty}}{[-\alpha q^3/ab, \alpha q^2/b, q/b, -\alpha q/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]$$

By using (2.10) in L.H.S. of (4.31), we get

$$(4.32) \frac{[\alpha q/ab; q]_{\infty} [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_{\infty}}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_{\infty}} = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_{\infty}}{[-\alpha q/ab, -\alpha q^3/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right]$$

$$-\frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_{\infty}}{[q/a, \alpha q^2/a; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right]$$

$$+\frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_{\infty}}{[-\alpha q^3/ab, -\alpha q/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -bq^2; q; \alpha q^2/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]$$

This is our main result (3.9)

Now by transforming I^{st} & III^{rd} ${}_2\Psi_2$ with the help of (2.9) and II^{nd} ${}_2\Psi_2$ with the help of (2.8), in right side of (4.17), we get

$$(4.33) (1 - c/q)(1 - d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] = \frac{[az, bz, cq/abz, dq/abz; q]_{\infty}}{[q/a, q/b; c, d; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} abz/c, abz/d; q; cd/abz \\ az, bz \end{matrix} \right]$$

$$-\frac{c+d}{q} \frac{[azq, d/a, c/b, d/abz; q]_{\infty}}{[zq, d, q/b, cd/abzq; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, abzq/d; q; d/a \\ azq, c \end{matrix} \right]$$

$$+\frac{cd}{q^2} \frac{[azq^2, bzq^2, c/abzq, d/abzq; q]_{\infty}}{[q/a, q/b, c, d; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} abzq^2/c, abzq^2/d; q; cd/abzq^2 \\ azq^2, bzq^2 \end{matrix} \right]$$

Taking $c = \alpha q^2/a, d = \alpha q^2/b$, & $z = -\alpha q/ab$, we get

$$(4.34) (1 - \alpha q/a)(1 - \alpha q/b) {}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_{\infty}}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right]$$

$$-\frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_{\infty}}{[-\alpha q^2/ab, \alpha q^2/b, q/b, -\alpha q^2/ab; q]_{\infty}} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right]$$

$$+ \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]$$

By using (2.10) in L.H.S. of (4.34), we get

$$(4.35) \quad \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q^2]_\infty}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_\infty} = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, \alpha q^2/ab, \alpha q^2/ab, -q/b; q]_\infty}{[-\alpha q^2/ab, -\alpha q^2/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b; q; \alpha q^2/ab \\ -\alpha q^2/b, \alpha q^2/a \end{matrix} \right] \\ + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]$$

This is the main result (3.10)

Now by transforming the I^{st} & II^{nd} ${}_2\Psi_2$ with the help of (2.9) and III^{rd} ${}_2\Psi_2$ with the help of (2.8), in right side of (4.17) we get

$$(4.36) \quad (1 - c/q)(1 - d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] = \frac{[az, bz, cq/abz, dq/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abz/c, abz/d; q; cd/abz \\ az, bz \end{matrix} \right] \\ - \frac{c+d}{q} \frac{[azq, bzq, c/abz, d/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abzq/c, abzq/d; q; cd/abzq \\ azq, bzq \end{matrix} \right] \\ + \frac{cd}{q^2} \frac{[azq^2, d/a, c/b, d/abzq; q]_\infty}{[zq^2, d, q/b, cd/abzq^2; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abzq^2/d; q; cd/abzq^2 \\ azq^2, c \end{matrix} \right]$$

Taking $c = \alpha q^2/a, d = \alpha q^2/b$, & $z = -\alpha q/ab$, in (4.36), we get

$$(4.37) \quad (1 - \alpha q/a)(1 - \alpha q/b) {}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right] \\ + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_\infty}{[-\alpha q^3/ab, \alpha q^2/b, q/b, -\alpha q/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]$$

By using (2.10) in L.H.S. of (4.37), we get

$$(4.38) \quad \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a^2, \alpha q^2/b^2, q^2, \alpha q, q/\alpha; q]_\infty}{[\alpha q^2/a, q/a, -\alpha q/ab; q]_\infty} = \frac{[-\alpha q/b, -\alpha q/a, -q^2/a, -q^2/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a/q, -b/q; q; -\alpha q^3/ab \\ -\alpha q/b, -\alpha q/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_\infty}{[q/a, \alpha q^2/a; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right]$$

$$+ \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, \alpha q^2/ab, \alpha q^2/ab, -1/b; q]_\infty}{[-\alpha q^3/ab, -\alpha q/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -bq; q; \alpha q/ab \\ -\alpha q^3/b, \alpha q^2/a \end{matrix} \right]$$

This is our main result (3.11)

Now by transforming $I^{st} {}_2\Psi_2$ with the help of (2.8) and II^{nd} & $III^{rd} {}_2\Psi_2$ with the help of (2.9), in right side of (4.17), we get

$$(4.39) \quad (1 - c/q)(1 - d/q) {}_2\Psi_2 \left[\begin{matrix} a, b; q; z \\ c/q, d/q \end{matrix} \right] = \frac{[az, d/a, c/b, dq/abz; q]_\infty}{[z, d, q/b, cd/abz; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, abz/d; q; d/a \\ az, c \end{matrix} \right] \\ - \frac{c+d}{q} \frac{[azq, bzq, c/abz, d/abz; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abzq/c, abzq/d; q; cd/abzq \\ azq, bzq \end{matrix} \right] \\ + \frac{cd}{q^2} \frac{[azq^2, bzq^2, c/abzq, d/abzq; q]_\infty}{[q/a, q/b, c, d; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} abzq^2/c, abzq^2/d; q; cd/abzq^2 \\ azq^2, bzq^2 \end{matrix} \right]$$

Taking $c = \alpha q^2/a, d = \alpha q^2/b$, & $z = -\alpha q/ab$, in (4.39)

$$(4.40) \quad (1 - \alpha q/a)(1 - \alpha q/b) {}_2\Psi_2 \left[\begin{matrix} a, b; q; -\alpha q/ab \\ \alpha q/a, \alpha q/b \end{matrix} \right] = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_\infty}{[-\alpha q/ab, \alpha q^2/b, q/b, -\alpha q^3/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right] \\ + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, q/b, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]$$

By using (2.10) in L.H.S. of (4.40), we get

$$(4.41) \quad \frac{[\alpha q/ab; q]_\infty [\alpha q^2/a, \alpha q^2/b, q/a, -\alpha q/ab; q]_\infty}{[\alpha q^2/a, \alpha q^2/b, q/a, -\alpha q/ab; q]_\infty} = \frac{[-\alpha q/b, \alpha q^2/ab, \alpha q^2/ab, -q^2/b; q]_\infty}{[-\alpha q/ab, -\alpha q^3/ab; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} a, -b/q; q; \alpha q^2/ab \\ -\alpha q/b, \alpha q^2/a \end{matrix} \right] \\ - \frac{\alpha q(a+b)}{ab} \frac{[-\alpha q^2/b, -\alpha q^2/a, -q/a, -q/b; q]_\infty}{[q/a, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -a, -b; q; -\alpha q^2/ab \\ -\alpha q^2/b, -\alpha q^2/a \end{matrix} \right] \\ + \frac{\alpha^2 q^2}{ab} \frac{[-\alpha q^3/b, -\alpha q^3/a, -1/a, -1/b; q]_\infty}{[q/a, \alpha q^2/a, \alpha q^2/b; q]_\infty} {}_2\Psi_2 \left[\begin{matrix} -\alpha q, -bq; q; -\alpha q/ab \\ -\alpha q^3/b, -\alpha q^3/a \end{matrix} \right]$$

This is our main result (3.12).

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